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On some algorithmic investigations of star partitions of graphs

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Abstract

Star partitions of graphs were introduced in a recent paper by the same authors in order to extend spectral methods in algebraic graph theory. Here it is shown that the corresponding partitioning problem is polynomial. Two algorithms are investigated: the first is based on the maximum matching problem for graphs, and the second invokes an algorithm for matroid intersection.

1. Introduction

Let G be a graph with vertices $1, \ldots, n$ and (0, 1)-adjacency matrix A. Let $\mu_1, \ldots, \mu_m$ ($\mu_1 > \cdots > \mu_m$) be the distinct eigenvalues of A, with corresponding eigenspaces $\mathscr{E}(\mu_1), \ldots, \mathscr{E}(\mu_m)$. For each $i \in \{1, \ldots, m\}$, let k_i be the multiplicity of μ_i, and let E_i be an $n \times k_i$ matrix whose columns are the vectors of some basis of $\mathscr{E}(\mu_i)$. Any matrix of the form

$$E = [E_1 | E_2 | \cdots | E_m] \tag{1.1}$$

is called an *eigenvector matrix* for G. The rows of E are indexed by the vertices of G and so any permutation of rows of E induces the same permutation of the vertices of G, and vice versa.

Let

$$A = \mu_1 P_1 + \cdots + \mu_m P_m$$

be the usual spectral decomposition of A. Thus P_i represents the orthogonal projection onto $\mathscr{E}(\mu_i)$ and, if $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$, the vectors $P_i\boldsymbol{e}_1, \ldots, P_i\boldsymbol{e}_n$ constitute a *eutactic star* in the sense of [13].

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A partition $X_1 \dot{\cup} \cdots \dot{\cup} X_m$ of the vertex set $\{1, \ldots, n\}$ is called a *star partition*, with *star cells* $X_1, \ldots, X_m$, if for each $i \in \{1, \ldots, m\}$ the vectors $P_i \boldsymbol{e}_j$ $(j \in X_i)$ are linearly independent. In this situation a comparison of dimensions shows that $|X_i| = k_i$ $(i = 1, \ldots, m)$ and the vectors $P_i \boldsymbol{e}_j$ $(j \in X_i)$ form a basis $\mathscr{B}_i$ of $\mathscr{E}(\mu_i)$. Then $\mathscr{B}_1 \cup \cdots \cup \mathscr{B}_m$ is a basis of $\mathbb{R}^n$, called in [6] a *star basis* corresponding to A (a construction applicable to any symmetric matrix with real entries). Any partition $X_1 \dot{\cup} \cdots \dot{\cup} X_m$ of $\{1, \ldots, n\}$ with the property that $|X_i| = k_i$ $(i = 1, \ldots, m)$ is called a *feasible* partition.

It was shown in [6] that every graph G has a star partition; moreover the partition $X_1 \dot{\cup} \cdots \dot{\cup} X_m$ is a star partition if and only if for each $i \in \{1, \ldots, m\}$, μ_i is not an eigenvalue of $G - X_i$ (the subgraph induced by the complement of X_i in $\{1, \ldots, n\}$). For later reference we outline here a variant of the existence proof [12]. Let $\{\boldsymbol{x}_h\colon h \in R_i\}$ be an arbitrary fixed basis of $\mathscr{E}(\mu_i)$, with $R_1 \dot{\cup} \cdots \dot{\cup} R_m = \{1, \ldots, n\}$; and let

$$\boldsymbol{e}_j = \sum_{h=1}^{n} t_{hj} \boldsymbol{x}_h \quad (j = 1, \ldots, n). \tag{1.2}$$

Since the transition matrix (t_{hj}) is invertible there exists a feasible partition $X_1 \dot{\cup} \cdots \dot{\cup} X_m$ of $\{1, \ldots, n\}$ such that the determinant of each $k_i \times k_i$ matrix (t_{hj}) $((h, j) \in R_i \times X_i)$ is non-zero. (To see this, consider the multiple Laplacian development of $\det(t_{hj})$ corresponding to the fixed row partition determined by $R_1, \ldots, R_m$.) Since $P_i \boldsymbol{e}_j = \sum_{h \in R_i} t_{hj} \boldsymbol{x}_h$ it follows that the vectors $P_i \boldsymbol{e}_j (j \in X_i)$ are linearly independent.

Star partitions of a graph are used in [6] to construct a star basis which characterizes the graph. Accordingly an algorithm for the construction of a star basis is required for practical purposes; moreover such an algorithm should be polynomial if a worthwhile reduction of the graph isomorphism problem is to be achieved. In Section 3 we give an explicit polynomial algorithm for finding a star partition of an arbitrary graph, and in Section 4 we note that the existence of a polynomial algorithm is already implicit in a result of Edmonds [8] on matroid intersection. In Section 5 we add a few remarks on the enumeration of star partitions.

We shall require the notion of the *König digraph* of a matrix $W = (w_{ij})_{m \times n}$: this is the weighted bipartite digraph $K(W) = (V_1, V_2, E; w)$, where $V_1 = \{1, \ldots, m\}$, $V_2 = \{1, \ldots, n\}$, $E = \{ij \in V_1 \times V_2\colon w_{ij} \neq 0\}$ and w is the weight function $w\colon E \to \mathbb{R}$ given by $w(ij) = w_{ij}$. The term "König digraph" was introduced in [3] in view of König's use of the digraph in investigating certain problems in matrix theory [10]. The present article provides a further example of the symbiotic relationship between combinatorics and matrix theory discussed by Brualdi [2]: matrix theory is applied to graphs (cf. [5]) and graph theory to matrices (cf. [3]). More precisely, in the construction of a star partition in Section 3, we associate with a graph G an adjacency matrix, form the König digraph $K(E)$ of a corresponding eigenvector matrix E, and reduce our problem to a matching problem for $K(E)$.

2. Preliminaries

We shall require a further characterization of star partitions of a graph G:

Theorem 2.1. *The partition $X_1 \dot{\cup} \cdots \dot{\cup} X_m$ of the vertex set of G is a star partition for G if and only if an eigenvector matrix (after an appropriate choice of eigenvectors and an appropriate ordering of the vertices of G) has the form E^* given below:*

$$E^* = \begin{array}{c} \begin{array}{cccc} \mathscr{E}(\mu_1) & \mathscr{E}(\mu_2) & \cdots & \mathscr{E}(\mu_m) \end{array} \\ \left[\begin{array}{cccc} I_{k_1} & * & & * \\ & I_{k_2} & & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & & I_{k_m} \end{array}\right] \end{array} \begin{array}{c} \\ X_1 \\ X_2 \\ \vdots \\ X_m \end{array} \tag{2.1}$$

In the matrix (2.1), $\mathscr{E}(\mu_i)$ designates the columns that correspond to this eigenspace, X_i designates the rows that correspond to this star cell, while $*$ denotes a block matrix of an appropriate size and I_k denotes the identity matrix of order k.

Proof. The form (2.1) was first established in [6, Section 2], while [12, Proposition 3.3] shows that the existence of such a matrix E^* is not only necessary but sufficient for $X_1 \dot{\cup} \cdots \dot{\cup} X_m$ to be a star partition. □

Remark 2.2. In [6, cf. Theorem 2.7] all entries of the matrix E^* from (2.1) are found explicitly.

Remark 2.3. The statement of Theorem 2.1 remains true if instead of the unit matrices on the main diagonal we have invertible matrices of the appropriate sizes, i.e. if I_{k_i} is replaced by an invertible matrix D_i for each i ($i = 1, \ldots, m$).

Remark 2.4. The eigenvector matrix given by (2.1) has an interesting combinatorial interpretation from the standpoint of its König digraph (see Fig. 1).

For each $i \in \{1, \ldots, m\}$ the subgraph of $K(E^*)$ induced by vertices corresponding to $\mathscr{E}(\mu_i)$ and X_i represents part of a perfect matching (that is, it consists of k_i copies of

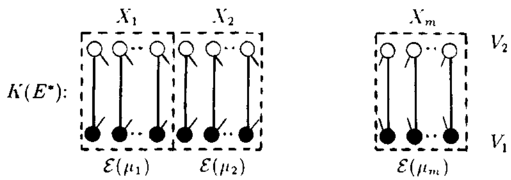


Fig. 1.

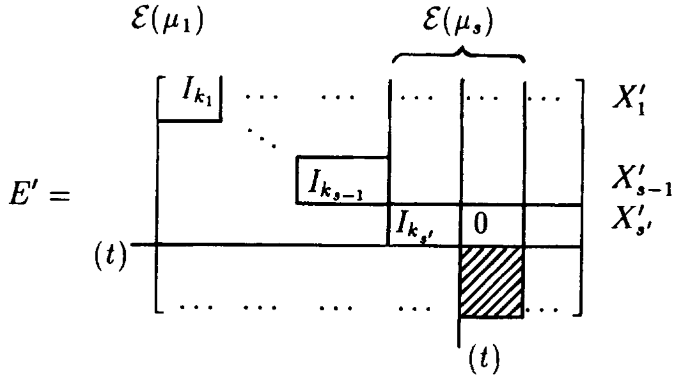
arcs of weight 1). All other arcs join vertices corresponding to $\mathscr{E}(\mu_i)$ and X_j $(i \neq j)$. In the next section we shall see some advantages of this interpretation.

3. An algorithm for star partitions

We give an explicit algorithm for constructing a star partition of a given graph, based on Theorem 2.1. The rough idea of the algorithm is as follows.

Start with any eigenvector matrix, as given by (1.1), and by means of elementary column operations within each block E_i $(i = 1, \ldots, m)$, together with row permutations, transform it to the form (2.1).

We show that there exists a good strategy (with polynomial time bound) for this construction. This (almost greedy) strategy is as follows. For an instance graph, take an (arbitrary) eigenvector matrix, say E (as given by (1.1)). Suppose next that we have somehow, for some s $(1 \leqslant s \leqslant m)$, transformed the blocks $E_1, \ldots, E_{s-1}$ to the forms in (2.1), but have failed to do the same for the block E_s. In addition, suppose that only one part of E_s (a block of size $k_{s'} \times k_s$; $k_{s'} < k_s$) is transformed as required. For more details see the matrix E', given by (3.1). The reason for failure is the absence of a non-zero element in the shaded area of E', so that no pivot can be brought to the position (t, t), where $t = k_1 + \cdots + k_{s-1} + k_{s'} + 1$. Otherwise, if we can find a pivot in the shaded part of E', we can easily augment E'.



(3.1)

In graph-theoretic terms, we have encountered the following situation: all sets $X'_1, \ldots, X'_{s-1}$ so far formed are putative star cells, while $X'_{s'}$ (the sth partial cell) is not, and cannot be extended to a star cell using the vertices which are not yet classified. Thus at this point we need to make some adjustments, and thereafter extend $X'_{s'}$ by at least one vertex.

To this end, suppose first that we can find in the matrix E' a sequence of non-diagonal non-zero entries, say $\alpha, \beta, \ldots, \tau$, whose positions can be most conveniently visualized as in Fig. 2. Notice that each entry ν of the sequence is reachable from the previous entry μ in the sequence by moving vertically from μ towards the main diagonal (the entry μ^*), and then moving horizontally towards ν. Also notice that τ is in the tth column while α is in the uth row for some $u \geqslant t$. Assume now that the rows of E' which contain the entries from the above sequence are permuted cyclically according to the rule: for each entry μ, the row containing μ replaces the row containing μ^* (to visualize this, see Fig. 2). Consequently, the non-zero entries $\alpha, \beta, \ldots, \tau$ are now moved to diagonal positions (those of entries $\alpha^*, \beta^*, \ldots, \tau^*$, respectively) and accordingly we have a pivot at position (t, t). The problem we might encounter now concerns the blocks along the main diagonal: some of the s unit matrices (including $I_{k_{s'}}$) may be destroyed. We can recover them by means of elementary column operations within the corresponding blocks E_i if and only if these blocks still have full rank. In order to show that this is indeed the case for an appropriate choice of $\alpha, \beta, \ldots, \tau$ we make use of some well-known tools from matching theory (see, for example [11]).

Let us first interpret the matrix E' in terms of the corresponding König digraph (see Fig. 3). Notice that there are two kinds of arcs in this graph. In particular, the boldface arcs correspond to diagonal entries (up to position $t - 1$) and form a matching (to be denoted by M'). Also notice that the vertices in V_1 (resp. V_2) corresponding to columns (resp. rows) are coloured black (resp. white). The black vertex t is not adjacent to any white vertex t' ($t' \geqslant t$), since otherwise we can easily introduce a pivot at position (t, t). Considering again the sequence $\alpha, \beta, \ldots, \tau$ we observe that it corresponds to a matching in $K(E')$. On the other hand, the sequence $\alpha, \alpha^*, \beta, \beta^*, \ldots, \sigma, \sigma^*, \tau$ corresponds to an *augmenting path* with respect to M'. Recall that an augmenting path (for which orientation of arcs is ignored) with respect to some matching M is a path whose endvertices are both *free* (i.e. are not incident to edges from M) and whose

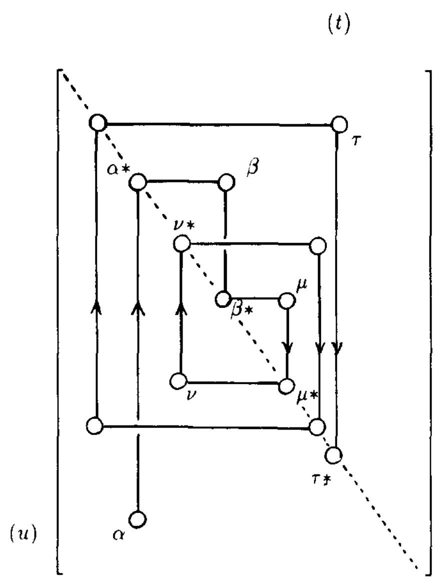


Fig. 2.

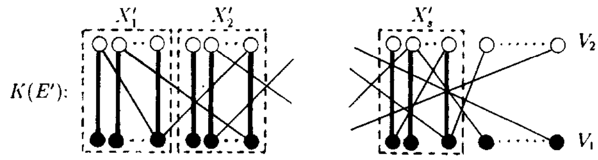


Fig. 3.

edges are alternatively in $\bar{M}$ and M. (Here $\bar{M}$ denotes the complement of M with respect to the edge set of the digraph.) The following lemma establishes the existence of a sequence of the type $\alpha, \beta, \dots, \tau$ described above.

Lemma 3.1. *Given the matching M' in $K(E')$, let t be the vertex in V_1 as above (black and free). There exists an augmenting path (with respect to M') with t as an endvertex.*

Proof. The matrix E' has full rank n because $\det(E') \neq 0$. Thus $K(E')$ admits a perfect matching N (of cardinality n). Let K be a subset $\{u_i v_i\colon i = 1, \dots, k\}$ of N defined by:

(a) $u_1 = t$ (and hence $v_1 < t$);

(b) $u_{i+1} = v_i$ $(1 \leqslant i \leqslant k-1)$;

(c) $v_i < t$ $(1 \leqslant i \leqslant k-1)$ and $v_k \geqslant t$.

Thus $u_i \leqslant t$ for each i, while $v_i < t$ for each $i \neq k$; also notice that $u_i \neq v_i$ for each i. Now consider the graph $H = (V_1, V_2, K \cup M')$: the component of H containing the vertex $t \in V_1$ is an augmenting path as required. □

Remark 3.2. A famous theorem of Berge [1] guarantees the existence of an augmenting path (with respect to M'), but not necessarily one with a prescribed endvertex.

Remark 3.3. The existence of a sequence of non-zero non-diagonal entries $\alpha, \beta, \dots, \tau$ as specified in Fig. 2 can also be established as follows, where $(e_{ij}) = E'$. Since $\det(E') \neq 0$, there is a permutation σ of $\{1, 2, \dots, n\}$ such that $\prod_{j=1}^{n} e_{\sigma(j)j} \neq 0$. Then $\sigma(t) < t$, $\sigma^{i+1}(t) \neq \sigma^i(t)$ for all i and there exists a positive integer h such that $\sigma^h(t) = t$. Let k be the least positive integer such that $\sigma^k(t) \geqslant t$. Now we may take $\alpha = e_{\sigma^k(t)\sigma^{k-1}(t)}$, $\beta = e_{\sigma^{k-1}(t)\sigma^{k-2}(t)}, \dots, \tau = e_{t\sigma(t)}$. Thus $u = \sigma^k(t)$ and a suitable permutation of columns is obtained when $\sigma^i(t)$ replaces $\sigma^{i-1}(t)$ $(i = 1, \dots, k)$.

Lemma 3.4. *Any shortest augmenting path starting at the vertex $t \in V_1$ enables the matrix E' to be augmented.*

Proof. By Lemma 3.1, there is an augmenting path with respect to M', say P, which starts at $t \in V_1$. Then the symmetric difference $M' \Delta P$ is a matching in $K(E')$, and $|M' \Delta P| = |M'| + 1$. From our earlier considerations, this yields an augmentation of E' only if the new blocks along the main diagonal (including the sth, i.e. the one being extended) have full rank. We will now show that this is indeed true provided P has shortest length. To prove this claim, let M_i $(1 \leqslant i \leqslant s)$ be the part of M' corresponding to the ith diagonal block. (Equivalently the arcs from M_i are incident only with the vertices corresponding to $\mathscr{E}(\mu_i)$ and X_i'). Let $g_1, \dots, g_r$ be the arcs of $P \cap M_i$ ordered as they are encountered on traversing P from the endvertex $t \in V_1$ to the endvertex $u \in V_2$, and let $g_1', \dots, g_r'$ be the arcs of P immediately prior to $g_1, \dots, g_r$, respectively; see Fig. 4. Now notice that w_1 is not adjacent to $u_2, \dots, u_r$; w_2 is not adjacent to $u_3, \dots, u_r$; and so on, since P is a shortest path. After augmentation by P, M_i becomes M_i', where g_i is replaced by g_i' for each i. Consider now the subgraph of $K(E')$ induced by the vertex set of M_i' $(1 \leqslant i \leqslant s)$. This subgraph corresponds to a (lower) triangular matrix with non-zero diagonal entries. □

Theorem 3.5. *There exists a polynomial-time algorithm for finding one star partition of any graph.*

Proof. Suppose that our instance graph has n vertices. As is well known from linear algebra, an eigenvector matrix can be obtained in polynomial time with respect to n. By [7], the complexity is at worst $O(n^3)$. Of course, we suppose here that enough decimal places are taken in representing the real numbers in question, in order to have control over numerical calculations and comparisons. The important point in this respect is the fact that always we have to make only zero versus non-zero decisions.

Now starting from any eigenvector matrix as above, we can easily construct the corresponding König digraph. Thereafter we are faced rather with combinatorial problems. In each step we have repeatedly (n times) to carry out the following:

(a) find an augmentation path (with prescribed endvertex) in the current König digraph;

(b) update the structure of this digraph to match the diagonal pattern of the eigenvector matrix (after a specified permutation of rows).

Hopcroft and Karp [9] provide a polynomial algorithm for finding a shortest augmenting path beginning at a prescribed vertex t of V_1: what is required is step 1 of their Algorithm A applied to the graph obtained from the König digraph by deleting the free vertices in $V_1 \backslash \{t\}$. This has complexity $O(n^2)$, while our task (b) has complexity $O(n^3)$ at worst. Accordingly our algorithm is polynomial with complexity $O(n^4)$ at worst. □

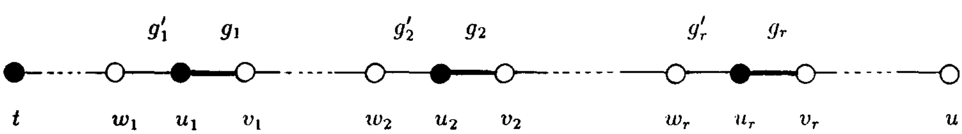


Fig. 4.

We conclude this section with some observations on the construction of further star partitions by exchanging elements between cells of a given partition.

Proposition 3.6. *Suppose that* $X_1 \dot\cup \cdots \dot\cup X_m$ *is a star partition and* $Y_1 \dot\cup \cdots \dot\cup Y_m$ *is a feasible partition for G. Let* $S_i = Y_i \backslash X_i$, $T_i = X_i \backslash Y_i$ *and let*

$$E^*\binom{S_i}{T_i}$$

be the submatrix of an eigenvector matrix (2.1) *whose rows are indexed by* S_i *and whose columns are indexed by* T_i $(i = 1, \ldots, m)$. *Then* $Y_1 \dot\cup \cdots \dot\cup Y_m$ *is a star partition if and only if*

$$\prod_{i=1}^{m} \det E^*\binom{S_i}{T_i} \neq 0.$$

(Here

$$\det E^*\binom{S_i}{T_i}$$

is interpreted as 1 *if* $S_i = T_i = \emptyset$.)

Proof. This follows from Remark 2.3 since

$$E^*\binom{S_i}{T_i}$$

is the ith diagonal block in an eigenvector matrix obtained by substituting rows indexed by S_i for those indexed by T_i $(i = 1, \ldots, m)$. □

Proposition 3.7. *Let G be a bipartite graph whose non-zero eigenvalues include* μ *and* $-\mu$, *and let* X^+, X^- *be the cells of a star partition which correspond to* μ, $-\mu$, *respectively. We may interchange* X^+ *and* X^- *to obtain a second star partition in which* X^+, X^- *correspond to* $-\mu$, μ, *respectively.*

Proof. The effect of the interchange on an eigenvector matrix E^* is to replace a pair of invertible principal submatrices of E^* with another pair differing from the original only in the signs of certain rows. □

We note also an application of Proposition 3.6 in the special case that $|S_i| = |T_i| \leqslant 1$ for each i. If $E^* = (e_{ij})$ then there exists $\sigma \in S_n$ such that $\prod_{j=1}^{n} e_{\sigma(j)j} \neq 0$. Suppose that σ has a constituent cycle $\mu = (j_1, j_2, \ldots, j_k)$ such that the elements $j_1, \ldots, j_k$ lie in k different cells of a star partition. Since $e_{\sigma(j)j} \neq 0$ for all j we can generate a second star partition by substituting $\mu(j_h)$ for j_h in the relevant star cell for each $h \in \{1, \ldots, k\}$.

4. An alternative approach

As noted in [11] the theory of matroids underlies the construction of matchings such as that related to the eigenvector matrix E^* in Section 3. The matrix E^* is the transition matrix from the standard basis of $\mathbb{R}^n$ to a basis of eigenvectors, and its inverse may be taken to be the transition matrix (t_{hj}) of Eq. (1.2). In this section we work with (t_{hj}) and appeal to matroid theory to establish the existence of a polynomial algorithm of complexity at most $O(n^5)$ which enables the cells of a star partition to be constructed in succession without backtracking; that is, once $X_1, \dots, X_s$ $(s < m)$ are constructed they are not subject to subsequent modification.

Recall that $\boldsymbol{e}_j = \sum_{h=1}^{n} t_{hj}\boldsymbol{x}_h$ where $\{\boldsymbol{x}_h : h \in R_i\}$ is a basis for $\mathscr{E}(\mu_i)$ $(i = 1, \dots, m)$. Here we take $k = k_1$, $R_1 = \{1, \dots, k\}$ and we write the columns of (t_{hj}) as

$$\begin{pmatrix} \boldsymbol{a}_j \\ \boldsymbol{b}_j \end{pmatrix} \quad (j = 1, \dots, n),$$

where $\boldsymbol{a}_j = (t_{1j}, \dots, t_{kj})^{\mathrm{T}}$. The Laplacian expansion of $\det(t_{hj})$ determined by R_1 guarantees the existence of a k-subset X_1 of $\{1, \dots, n\}$ such that both the $k \times k$ matrix (t_{hj}) $(h \in R_1, j \in X_1)$ and the $(n-k) \times (n-k)$ matrix (t_{hj}) $(h \notin R_1, j \notin X_1)$ are invertible. Thus X_1 is an independent set of greatest size in the intersection of two matroids: one is the linear matroid determined by the vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$, and the other is the dual of the linear matroid determined by the vectors $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n$. For this observation the authors are indebted to M.R. Jerrum, who drew attention to a result of Edmonds [8] on matroid intersection. Edmonds gives an explicit algorithm for finding a set of maximal cardinality which is independent in each of two matroids on a given finite set; moreover the algorithm is polynomial when for each of the two matroids there is a polynomial algorithm for determining whether a given subset is independent. In our context this last condition is satisfied since there exists a polynomial algorithm for finding the dimension of a subspace spanned by a finite set of vectors. Accordingly X_1 can be found in polynomial time. To find X_2, we apply the same process to the $(n-k) \times (n-k)$ matrix (t_{hj}) $(h \notin R_1,\ j \notin X_1)$. Repetition for each of $R_3, \dots, R_{m-1}$ yields a star partition $X_1 \dot{\cup} \cdots \dot{\cup} X_m$. Each application of Edmonds' algorithm requires at most $O(n^4)$ steps and so the star partition is obtained in at most $O(n^5)$ steps.

Proposition 4.1. *For any vertex v of a connected graph G, there exists a star partition of G in which $\{v\}$ is the cell corresponding to the largest eigenvalue of G.*

Proof. With the notation of this section, let μ_1 be the largest eigenvalue of G. Since G is connected, μ_1 is a simple eigenvalue (with corresponding eigenvector $\boldsymbol{x}_1$) and $t_{1j} = \boldsymbol{x}_1 \cdot \boldsymbol{e}_j \neq 0$ for all $j \in \{1, \dots, n\}$ [5]. Without loss of generality, $v = 1$ and it suffices to show that the matrix with entries t_{hj} $(h > 1, j > 1)$ is invertible. Now

$$(I - P_1)\boldsymbol{e}_j = \sum_{h=2}^{n} t_{hj}\boldsymbol{x}_h \quad (j = 2, \dots, n)$$

and the orthogonal projection $I - P_1$ has kernel $\langle x_1 \rangle$, which has trivial intersection with $\langle e_2, \ldots, e_n \rangle$ because $x_1 \cdot e_1 \neq 0$. Accordingly the matrix (t_{hj}) $(h > 1, j > 1)$ represents an invertible linear transformation. □

5. Some enumerative considerations

It is known that any graph has at least one star partition. Let $SP(G)$ denote the number of star partitions of a graph G. For example, it is easy to see that $SP(\bar{K}_n) = 1$, and $SP(K_n) = n$, whereas $SP(K_{m,n}) = 2mn$.

Proposition 5.1. *For a graph G on n vertices with eigenvalue multiplicities $k_1, \ldots, k_m$, we have*

$$1 \leqslant SP(G) \leqslant \frac{n!}{k_1! \ldots k_m!}.$$

Proof. It is easy to see that the upper bound represents the number of (ordered) feasible partitions of G. □

In particular, if all eigenvalues are simple then we have $SP(G) \leqslant n!$. We shall see (Proposition 5.3) that there exists an infinite family of graphs for which this bound is attained.

Proposition 5.2. *We have $SP(G) = 1$ if and only if $G = \bar{K}_n$ for some n.*

Proof. The only graph G with just one distinct eigenvalue is $\bar{K}_n$, for which $SP(G) = 1$. If G has at least two distinct eigenvalues then the same is true of a component C of G, and we have $SP(C) > 1$ by Proposition 4.1. Given a star partition of each component we obtain a star partition of G by combining star cells corresponding to the same eigenvalue [6, Theorem 3.12]. It follows that $SP(G) > 1$. □

Two star partitions of a graph G are called *isomorphic* if there is an automorphism of G taking one partition to another. Let $NSP(G)$ denote the number of non-isomorphic star partitions of G. Of course, $NSP(G) \leqslant SP(G)$. For example, $NSP(K_n) = 1$; $NSP(K_{m,n}) = 2$ for $m \neq n$ and $NSP(K_{n,n}) = 1$. In addition, if G is the Petersen graph then $NSP(G) = 10$ while $SP(G) = 750$.

Proposition 5.3. *If G is a path on n vertices, then $SP(G) = n!$ if and only if $n + 1$ is a prime number.*

Proof. Since $G = P_n$ for some n, the eigenvalues of G are simple and also $\mu_i = 2\cos \pi i/(n + 1)$ $(i = 1, \ldots, n)$. From [6, Theorems 3.9, 3.11] $SP(G) = n!$ if and only if μ_i is not an eigenvalue of $G - k$, for any $k = 1, \ldots, n$. Since $G - k$ is a graph with each component a path, the former claim is true if and only if

$$2\cos\frac{\pi i}{n+1} \neq 2\cos\frac{\pi j}{m+1}$$

for each m, i and j ($1 \leqslant m < n$; $i = 1, \ldots, n$; $j = 1, \ldots, m$). The latter is possible if and only if $n + 1$ is a prime number. □

Remark 5.4. If $n + 1$ ($n \neq 1$) is prime then $NSP(P_n) = n!/2$ since P_n has only one non-trivial automorphism when $n > 1$.

Remark 5.5. If G is a cycle of length at least 4 then (by the same argument as in the proof of Proposition 5.3) at least one feasible partition of G is not a star partition.

For graphs having only simple eigenvalues we have the following necessary and sufficient condition for every feasible partition to be a star partition.

Proposition 5.6. *For a graph G all of whose eigenvalues are simple, any feasible partition of G is a star partition if and only if all entries of an eigenvector matrix are non-zero.*

Proof. Note that all eigenvector matrices have the same pattern of zeros. The result follows immediately from Theorem 2.1. □

Example. For an eigenvector matrix of P_n we may take the matrix $[\sin \pi ij/(n+1)]_{n \times n}$ (see, e.g. [5, p. 214]). On applying Proposition 5.6 we obtain another proof of Proposition 5.3.

More generally, for a graph having only simple eigenvalues, the problem of enumerating the star partitions becomes equivalent to the problem of enumerating the perfect matchings in the associated König digraph. For in an $n \times n$ eigenvector matrix E we need to find n non-zero entries with exactly one in each row and column. This is precisely a perfect matching in $K(E)$ and so $SP(G)$ becomes equal to the number of perfect matchings in $K(E)$. Enumeration of perfect matchings in a bipartite graph can be performed in several ways (see, e.g. [11]).

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